© INTERNATIONAL JOURNAL FOR RESEARCH PUBLICATION & SEMINAR
ISSN: 2278-6848 | Volume: 05 Issue: 01 | March 2014
Paper is available at www.jrps.in | Email : info@jrps.in

ROUGH *k*-IDEALS IN SEMIRINGS

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Abstract

In this paper we introduce the notions of k-ideals and k-closure in semirings. We have shown that I is a k-ideal of a semiring R if and only if it is a rough k-ideal of R. We have also shown that if I and J are left(right) ideals of a emiring R then $(\theta(I):\theta(J))$ is a rough ideal of R.

Keywords: Semirings; congruence relation; k-ideals; k-closure; lower approximation; upper approximation

ISSN : 2278-6848
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1 Introduction

The theory of rough set was first introduced by Pawlak[9,10]. Rough set theory deals with inexact, uncertain or vague knowledge. It has recently been fascinated by researchers who work in both real life applications and theoretical developments. Rough set theory is an extension of set theory. It has many practical applications in areas such as data classification, data analysis, machine learning and knowledge discovery. Biswas and Nanda [3] introduced the notion of rough subgroups. Kuroki [7] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [8] gave some properties of the lower and upper approximations in groups. Davvaz [5] gave the relationship between rough set and ring theory. He considered a ring as the universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal. Thillaigovindan et.al. [11,12] have studied the relationship between rough set theory and near ring.

This paper consists of 4 sections. In section 2 we give the basic definitions and results which are essential for the development of the new results. In section 3 we introduce rough k-ideals and rough closure in semirings. We have shown that I is a k-ideal of a semiring R iff it is a rough k-ideal of R. We have also shown that if I and J are left(right) ideals of a semiring R then $(\theta(I):\theta(J))$ is a rough ideal of R. A brief conclusion is given in section 4.

2 Preliminaries and Congruence Relation

In this section some definitions and results are reproduced, which are proposed by pioneers in this field earlier and are necessary for the development of the new results.

A *semiring* is a system consisting of a nonempty set R together with binary operations on R called *addition* and *multiplication* such that $(R,+)$ is a semigroup; $(R,.)$ is a semigroup and $a.(b+c)=a.b+a.c$ and $(b+c).a=b.a+c.a$ for all $a,b,c\in R$. A semiring R may have an *identity 1*,defined by $1.a=a=a.1$ and a zero 0, defined by $0+a=a=a+0$ and $a.0=0=0.a$ for all $a\in R$. From now on we write R for semirings. A nonempty subset I of R is said to be a *left*(resp. *right*) *ideal* if $x,y\in I$ and $r\in R$ imply that $x+y\in I$ and $r.x\in I$ and (resp. $x.r\in I$). I is said to be *two –sided ideal* or simply *ideal* of R, if I is both left and right ideal of R.

A left ideal I of a semiring R is said to be a *left k-ideal* if $a\in I, x\in R$ and $a+x\in I$ or $x+a\in I$, then $x\in I$. A *right k-ideal* is defined dually, and a *two sided k-ideal* or simply a *k-ideal* is both a left and a right k-ideal. The ideal generated by $a\in R$ is defined as the smallest ideal of R, which contains a and is denoted by $<a>$. The k-ideal generated by $a\in R$ is defined as the smallest k-ideal of R, which contains a and is denoted by $<a>_k$.

Let $a\in R$. We denote by $I(a)$ (resp., $L(a), R(a), L_k(a), R_k(a)$) the *ideal*,(resp., *left ideal*, *right ideal*, *left k-ideal*, *right k-ideal*) of R generated by $a\in R$. One can easily prove that

$$I(a)=\{a+sa+at+s_iat_i|\, s,t,s_i,t_i\in R\}\,;\, L(a)=\{a+sa|\, s\in R\}\,;\, R(a)=\{a+at|\, t\in R\}$$

$L_k(a) = \{u \in R | u + a + sa = a + sa \; for \; some \; s \in R\}$
$R_k(a) = \{u \in R | u + a + sa = a + as \; for \; some \; s \in R\}$
For the sake of convenience we write ab instead of $a \cdot b$.

Let U be a universal set. An *equivalence relation* θ on R is a reflexive, symmetric and transitive binary relation on U. The set of elements of U that are related to $x \in U$ is called the equivalence class of x and is denoted by $[x]_\theta$.

Definition 2.1 A pair (U, θ) wher $U \neq \phi$ and θ is an equivalence relation on U, is called an *approximation space*

Definition 2.2 For an approximation space (U, θ) by a *rough approximation* in (U, θ) we mean a mapping $\rho: \wp(U) \to \wp(U) \times \wp(U)$ defined as $\rho(X) = \left(\underline{\rho}(X), \overline{\rho}(X)\right)$ for $X \subseteq U$ where $\underline{\rho}(X) = \{x \in U | [x]_\theta \subseteq U\}$, and $\overline{\rho}(X) = \{x \in U | [x]_\theta \cap U \neq \phi\}$, $\underline{\rho}(X)$ is called a *lower rough approximation* of X in (U, θ) where as $\overline{\rho}(X)$ is called an *upper rough approximation* of X in (U, θ).
Hereafter we use $\underline{\theta}(X), \overline{\theta}(X)$ and $\theta(X)$ instead of $\underline{\rho}(X), \overline{\rho}(X)$ and $\rho(X)$ respectively.

Definition 2.3 Given an approximation space (U, θ) a pair $(A, B) \in \wp(U) \times \wp(U)$ is called a *rough set* in (U, θ) if and only if $(A, B) = \theta(X)$ for some $X \subseteq U$.
If A nd B are any two subsets of R, then $AB = \{ab | a \in A, b \in B\}$.

Definition 2.4 Let θ be an equivalence relation on R. θ is called a *congruence relation* if $(a, b) \in \theta$ implies
(i) $(a + x, b + x) \in \theta$; (ii) $(x + a, x + b) \in \theta$; (iii) $(ax, bx) \in \theta$ and (iv) $(xa, xb) \in \theta$, for all $x \in R$.
The following theorem is an immediately consequence of Definition 2.4.

Theorem 2.5. *Let θ be a congruence relation on a semiring R.* Then $(a, b), (c, d) \in \theta$ implies $(a + c, b + d) \in \theta$, $(ac, bd) \in \theta$ for all $a, b, c, d \in R$.

Lemma 2.6. *Let θ be a congruence relation on R. If $a, b \in R$, then*
(i) $[a]_\theta + [b]_\theta \subseteq [a + b]_\theta$
(ii) $[a]_\theta . [b]_\theta \subseteq [ab]_\theta$

Proof. (i) Let $x \in R$. Suppose $x \in [a]_\theta + [b]_\theta$. Then there exist $y, z \in R$ such that $y \in [a]_\theta, z \in [b]_\theta$ and $x = y + z$. This means that $(a, y), (b, z) \in \theta$ and hence $(a + b, y + z) = (a + b, x) \in \theta$. Thus $x \in [a + b]_\theta$ and hence $[a]_\theta + [b]_\theta \subseteq [a + b]_\theta$.
(ii)Let $z = xy \in [a]_\theta . [b]_\theta$. Then $x \in [a]_\theta$ and $y \in [b]_\theta$. This implies that $(a, x) \in \theta$ and $(b, y) \in \theta$. Since θ is a congruence relation, $(ab, xy) \in \theta$.Thus $z = xy \in [ab]_\theta$ and hence $[a]_\theta . [b]_\theta \subseteq [ab]_\theta$.
A congruence relation θ on R is called complete if $[a]_\theta + [b]_\theta = [a + b]_\theta$ and $[a]_\theta . [b]_\theta = [ab]_\theta$.

Definition 2.7. Let θ be a congruence relation on R and A a subset of R. Then the sets $\underline{\theta}(A) = \{x \in R / [x]_\theta \subseteq A\}$ and $\overline{\theta}(A) = \{x \in S / [x]_\theta \cap A \neq \phi\}$ are called the lower and upper approximations of the set A, respectively. Let A be any subset of R. $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is called a rough set with repect to θ if $\underline{\theta}(A) \neq \overline{\theta}(A)$.

Lemma 2.8. *For any approximation space* (R, θ) and $P \subseteq R$, *the following hold:*
(i) $\underline{\theta}(R \backslash P) = R \backslash \overline{\theta}(P)$; (ii) $\overline{\theta}(R \backslash P) = R \backslash \underline{\theta}(P)$; (iii) $\overline{\theta}(P) = \left(\underline{\theta}(P^c)\right)^c$; (iv) $\underline{\theta}(P) = \left(\overline{\theta}(P^c)\right)^c$

Proof. The proof is obvious and hence omitted.
The following theorem is immediate.

Theorem 2.9. *Let θ and ψ be congruence relations on R and let A and B be nonempty subsets of R. Then*

(i) $\underline{\theta}(\mathrm{A}) \subseteq \mathrm{A} \subseteq \overline{\theta}(\mathrm{A})$

(ii) $\underline{\theta}(\emptyset) = \emptyset = \overline{\theta}(\emptyset)$

(iii) $\underline{\theta}(\mathrm{R}) = \mathrm{R} = \overline{\theta}(\mathrm{R})$

(iv) $\overline{\theta}(\mathrm{A} \cup \mathrm{B}) = \overline{\theta}(\mathrm{A}) \cup \overline{\theta}(\mathrm{B})$

(v) $\underline{\theta}(\mathrm{A} \cap \mathrm{B}) = \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$

(vi) $A \subseteq B$ implies $\underline{\theta}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{B})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\theta}(\mathrm{B})$

(vii) $\underline{\theta}(\mathrm{A} \cup \mathrm{B}) \supseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$

(viii) $\overline{\theta}(\mathrm{A} \cap \mathrm{B}) \subseteq \overline{\theta}(\mathrm{A}) \cap \overline{\theta}(\mathrm{B})$

(ix) $\theta \subseteq \psi$ implies $\underline{\psi}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\psi}(\mathrm{A})$

(x) $(\overline{\theta \cap \psi})(\mathrm{A}) = \overline{\theta}(\mathrm{A}) \cap \overline{\psi}(\mathrm{A})$

(xi) $(\underline{\theta \cap \psi})(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\psi)$

(xii) $\underline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$

(xiii) $\overline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$

(xiv) $\overline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$

(xv) $\underline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$.

Definition 2.10. Let A be any subset of R and (R, θ) be a rough approximation space. If $\underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A})$ are ideals , then $\underline{\theta}(\mathrm{A})$ is called a *lower and upper rough ideal* and $\overline{\theta}(\mathrm{A})$ is called an *upper rough ideal* of R, respectively. $\theta(A) = (\underline{\theta}(\mathrm{A}), \overline{\theta}(\mathrm{A}))$ is called *rough ideal* of R.

Theorem 2.11 *Let θ be a congruence relation on R. If A is a left (resp.right) ideal of R, then $\overline{\theta}(\mathrm{A})$ is a left (resp.right) ideal of R.*

Proof. Let $a, b \in \overline{\theta}(\mathrm{A})$. Then $[a]_\theta \cap A \neq \emptyset, [b]_\theta \cap A \neq \emptyset$. So there exist $x \in [a]_\theta \cap A$ and $y \in [b]_\theta \cap A$. Since $x, y \in A, x + y \in A$. Now $x + y \in [a]_\theta + [b]_\theta \subseteq [a + b]_\theta$. Therefore $[a + b]_\theta \cap A \neq \emptyset$ and this means that $a + b \in \overline{\theta}(\mathrm{A})$.

Again let $x \in \overline{\theta}(\mathrm{A})$ and $r \in R$. Then there exists $y \in [a]_\theta \cap A$ and $(y, x) \in \theta$. Since θ is congruence relation , $(xr, yr), (rx, ry) \in \theta$. This means that $xr, rx \in \overline{\theta}(\mathrm{A})$. Thus $\overline{\theta}(\mathrm{A})$ is an ideal of R.

Theorem 2.12. *Let θ be a congruence relation on R. If A is a left (resp.right) ideal of R and $\underline{\theta}(A)$ is nonempty, then $\underline{\theta}(A)$ is a left (resp.right) ideal of R.*

Proof. Let $a, b \in \underline{\theta}(\mathrm{A})$. Then $[a]_\theta, [b]_\theta \subseteq A$. Consider $[a + b]_\theta \subseteq [a]_\theta + [b]_\theta \subseteq A + A \subseteq A$. Thus $a + b \in \underline{\theta}(\mathrm{A})$.

Again let $a \in \underline{\theta}(\mathrm{A})$ and $r \in R$. Consider, $[ar]_\theta \subseteq [a]_\theta . [r]_\theta \subseteq AR \subseteq A$ and $[ra]_\theta \subseteq [r]_\theta . [a]_\theta \subseteq RA \subseteq A$. Thus $\underline{\theta}(\mathrm{A})$ is an ideal of R.

Corollary 2.13. *Let θ be a congruence relation on R. If A is an ideal of R and $\underline{\theta}(A)$ is nonempty, then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A)$ is a rough ideal of R.*

Lemma 2.14. *If I and J are ideals of R and $\underline{\theta}(I \cap J)$ is a nonempty set, then $\left(\underline{\theta}(I \cap J), \overline{\theta}(I \cap J)\right)$ is a rough ideal of R.*

Theorem 2.15. *Let φ be an epimorphism of a semiring R_1 to a semiring R_2 and let θ_2 be a congruence relation on R_2. Then*

(i) $\theta_1 = \{(a,b) \in R_1 \times R_2 | (\varphi(a), \varphi(b)) \in \theta_2\}$ *is a congruence relation.*

(ii) *If θ_2 is complete and φ is 1-1, then θ_1 is complete.*

(iii) $\varphi(\overline{\theta_1}(A)) = \overline{\theta_2}(\varphi(A))$

(iv) $\varphi(\underline{\theta_1}(A)) \subseteq \underline{\theta_2}(\varphi(A))$

(v) *If φ is 1-1 , then* $\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$

Proof. (i) Let $(a,b) \in \theta_1$ and $x \in R_1$. Then $(\varphi(a), \varphi(b)) \in \theta_2$. Since θ_2 is a congruence relation ,we have $\left(\varphi(a)+\varphi(x), \varphi(b)+\varphi(x)\right)$, $\left(\varphi(x)+\varphi(a), \varphi(x)+\varphi(b)\right)$, $\left(\varphi(a).\varphi(x), \varphi(b).\varphi(x)\right)$ and $\left(\varphi(x).\varphi(a), \varphi(x).\varphi(b)\right)$ are in θ_2. φ being homomorphism , $(\varphi(a+x), \varphi(b+x))$, $(\varphi(x+a), \varphi(x+b))$, $(\varphi(ax), \varphi(bx))$ and $(\varphi(xa), \varphi(xb))$ are in θ_2. Again since φ being onto, $(a+x, b+x)$, $(x+a, x+b)$, (ax, bx), $(xa, xb)$are in θ_1. Thus θ_1 is congruence relation in R_1.

(ii) Let θ_2 be complete . Assume that $z \in [ab]_{\theta_1}$. Then $(ab, z) \in \theta_1$. By definition of θ_2, $\left(\varphi(ab), \varphi(z)\right) \in \theta_2$. Hence

$$\begin{aligned} \varphi(z) &\in [\varphi(ab)]_{\theta_2} \\ &= [\varphi(a).\varphi(b)]_{\theta_2} \\ &= [\varphi(a)]_{\theta_2}.[\varphi(b)]_{\theta_2} \end{aligned}$$

Since $\varphi(z) \in [\varphi(a)]_{\theta_1}.[\varphi(b)]_{\theta_2}$, there exist $x, y \in R_1$ such that

$$\begin{aligned} \varphi(z) &= \varphi(x).\varphi(y) \\ &= \varphi(xy), \varphi(x) \in [\varphi(a)]_{\theta_2}, \varphi(y) \in [\varphi(b)]_{\theta_2} \end{aligned}$$

Since φ is 1-1 and by definition of θ_1, $z = xy$ and $x \in [a]_{\theta_1}$, $y \in [b]_{\theta_1}$. Thus $z \in [a]_{\theta_1}.[b]_{\theta_1}$ and therefore $[ab]_{\theta_1} \subseteq [a]_{\theta_1}.[b]_{\theta_1}$. By Lemma 2.6, $[a]_{\theta_1}.[b]_{\theta_1} \subseteq [ab]_{\theta_1}$. Hence $[ab]_{\theta_1} = [a]_{\theta_1}.[b]_{\theta_1}$. Again suppose $z \in [a+b]_{\theta_1}$.In the similar manner, one can get $[a+b]_{\theta_1} = [a]_{\theta_1} + [b]_{\theta_1}$. Thus θ_1 is complete.

(iii) Let $y \in \varphi(\overline{\theta_1}(A))$. Then there exists $x \in \overline{\theta_1}(A)$ such that $y = \varphi(x)$. This implies that $[x]_{\theta_1} \cap A \neq \emptyset$ and so there exists $a \in [x]_{\theta_1} \cap A$. Then $\varphi(a) \in \varphi(A)$ and $(a,x) \in \theta_1$ implies $(\varphi(a), \varphi(x)) \in \theta_2$. So $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Thus $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. This implies that $y = \varphi(x) \in \overline{\theta_2}(\varphi(A))$ and so

$$\varphi(\overline{\theta_1}(A)) \subseteq \overline{\theta_2}(\varphi(A)) \quad (1)$$

Again let $z \in \overline{\theta_2}(\varphi(A))$, there exists $x \in R_1$ such that $z = \varphi(x)$. Hence $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. So there exists $a \in A$ such that $\varphi(a) \in \varphi(A)$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. By definition of θ_1, we have $a \in [x]_{\theta_1}$. Thus $[x]_{\theta_1} \cap A \neq \emptyset$, which implies $x \in \overline{\theta_1}(A)$ and so $z = \varphi(x) \in \varphi(\overline{\theta_1}(A))$. It means that

$$\overline{\theta_2}(\varphi(A)) \subseteq \varphi(\overline{\theta_1}(A)) \quad (2)$$

From (1) and (2) the conclusion follows.

(iv) Let $y \in \varphi\left(\underline{\theta_1}(A)\right)$. Then there exists $x \in \underline{\theta_1}(A)$ such that $\varphi(x) = y$ and so we have $[x]_{\theta_1} \subseteq A$. Again let $b \in [y]_{\theta_2}$. Then there exists $a \in R_1$ such that $\varphi(a) = b$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Hence $a \in [x]_{\theta_1} \subseteq A$ and so $b = \varphi(a) \in \varphi(A)$. Thus $[y]_{\theta_2} \subseteq \varphi(A)$. This implies that

$y \in \underline{\theta_2}(\varphi(A))$ and so we have $\varphi\left(\underline{\theta_1}(A)\right) \subseteq \underline{\theta_2}(\varphi(A))$.

(v) Let $y \in \underline{\theta_2}(\varphi(A))$. Then there exists $x \in R_1$ such that $\varphi(x) = y$ and $[\varphi(x)]_{\theta_2} \subseteq \varphi(A)$. Let $a \in [x]_{\theta_1}$. Then $\varphi(a) \in [\varphi(x)]_{\theta_2}$ and so $a \in A$. Thus $[x]_{\theta_1} \subseteq A$ and $x \in \underline{\theta_1}(A)$. Hence $y \in \varphi(x) \in \varphi\left(\underline{\theta_1}(A)\right)$ and so we have $\underline{\theta_2}(\varphi(A)) \subseteq \varphi\left(\underline{\theta_1}(A)\right)$. By (iv), we have $\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$.

3 Rough *k*-ideals in semirings

In this section we introduce the notion of lower and upper approximations of k-ideals in semirings. We characterize the semirings through rough k-ideals in terms of lower and upper approximations. We also introduce the notion of rough k-ideal and k-closure in semirings.

Throughout the following discussion we assume that the lower and upper approximations $\underline{\theta}(A)$ of any subset A of R with repect to θ is nonempty unless otherwise stated.

For any nonempty subsets A, B of R, we write the set
$(A{:}B)_l = \{r \in R |\, rB \subseteq A\}$, $(A{:}B)_r = \{r \in R |\, Br \subseteq A\}$ and
$(A{:}B) = \{r \in R |\, rB \subseteq A \text{ and } Br \subseteq A\}$.

Definition 3.1. An ideal (left ideal, right ideal) I of R is called a *k-ideal* (*left k-ideal*, *right k-ideal*) if $a, a+b \in I$ implies $b \in I$ for any elements $a, b \in R$.

Definition 3.2. If A is an ideal of a semiring R, then $\hat{A} = \{a \in R | a + x \in A \text{ for some } x \in A\}$ is called *k-closure* of A.

Theorem 3.3. *Let θ be congruence relation on R and I be a subset of R. I is a k-ideal(resp. left k-ideal, right k- ideal) of R if and only if $\theta(I)$ is a rough k-ideal (resp. left k-ideal, right k- ideal) of R.*

Proof. Let I be a k-ideal of R. Let $a, a+b \in \overline{\theta}(I)$. Then $[a]_\theta \cap I = \emptyset$ and $[a+b]_\theta \cap I = \emptyset$. This implies that there exists $x, y \in R$ such that $x \in [a]_\theta, x \in I$ and $y \in [a+b]_\theta$ and $y \in I$. Now $x \in [a]_\theta$implies $a \in [x]_\theta \subseteq I, a \in I$ and $y \in [a+b]_\theta$ implies $a+b \in [y]_\theta \subseteq I, a+b \in I$. I being k-ideal, $b \in I$. Thus $[b]_\theta \cap I = \emptyset$ and this shows that $b \in \overline{\theta}(I)$ and thus $\overline{\theta}(I)$ k-ideal of R. Similarly, $\underline{\theta}(I)$ is a k-ideal of R. Thus $\theta(I)$ is a rough k-ideal of R.

Conversely, assume that $\theta(I)$ is rough k-ideal of R. Then both $\overline{\theta}(I)$ and $\underline{\theta}(I)$ are k-ideals of R. Suppose that $a, a+b \in I$,$a, b \in R$. Then $[a]_\theta \cap I = \emptyset$ and $[a+b]_\theta \cap I = \emptyset$. This means that $+b \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is a k–ideal of R, $b \in \overline{\theta}(I)$. Then there exists $x \in R$ such that $x \in I$ and $x \in [b]_\theta$. This implies that $b \in [x]_\theta \subseteq I$, hence we have $b \in I$. Thus I is a k-ideal of R.

Lemma 3.4. *Let θ be congruence relation on R and I be a subset of R. If I is an ideal(resp. left ideal, right ideal) of R, then $\hat{\theta}(I)$ is a rough k-ideal (resp. left ideal, right ideal) of R.*

Proof. By Corollary 2.13, $\theta(I)$ is rough ideal of R. Let a_1 and $a_2 \in R$. Suppose $a_1, a_2 \in \hat{\overline{\theta}}(I)$. Now $a_1 + x_1, a_2 + x_2 \in \overline{\theta}(I)$ for some $x_1, x_2 \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is an ideal of R,
$a_1 + a_2 + x_1 + x_2 \in \overline{\theta}(I)$ where $x_1 + x_2 \in \overline{\theta}(I)$ and so $a_1 + a_2 \in \hat{\overline{\theta}}(I)$. Again let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$.

For any $r \in R$, since $\overline{\theta}(I)$ is an ideal of R, $(a+x)r \in \overline{\theta}(I)$ and so $ar + xr \in \overline{\theta}(I)$ for some $xr \in \overline{\theta}(I)$. Thus $ar \in \hat{\overline{\theta}}(I)$. In a similar way, $ra \in \hat{\overline{\theta}}(I)$. Suppose $a, a+b \in \hat{\overline{\theta}}(I)$.
Then $a + x, (a+b) + y \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is an ideal , $\left((a+b)+y\right) + x \in \overline{\theta}(I)$ implies $b + ((a+x)+y) \in \overline{\theta}(I)$ and $(a+x)+y \in \overline{\theta}(I)$, hence we have $b \in \hat{\overline{\theta}}(I)$. Thus $\hat{\overline{\theta}}(I)$ is a k-ideal of R.

© INTERNATIONAL JOURNAL FOR RESEARCH PUBLICATION & SEMINAR
ISSN: 2278-6848 | Volume: 05 Issue: 01 | March 2014
Paper is available at www.jrps.in | Email : info@jrps.in

Lemma 3.5. *Let θ be congruence relation on R and A and B be any two subsets of R,*

(i) $\overline{\theta}\widehat{(A)\overline{\theta}}(B) = \hat{\overline{\theta}}\widehat{(A)\hat{\overline{\theta}}}(B)$.

(ii) $\underline{\theta}\widehat{(A)\,\underline{\theta}}(B) = \hat{\underline{\theta}}\widehat{(A)\,\hat{\underline{\theta}}}(B)$.

Proof. Since $\overline{\theta}(A) \subseteq \hat{\overline{\theta}}(A)$ and $\overline{\theta}(B) \subseteq \hat{\overline{\theta}}(B)$, $\overline{\theta}(A)\overline{\theta}(B) \subseteq \hat{\overline{\theta}}(A)\hat{\overline{\theta}}(B)$ and so $\overline{\theta}\widehat{(A)\overline{\theta}}(B) \subseteq \hat{\overline{\theta}}\widehat{(A)\hat{\overline{\theta}}}(B)$. Let $x \in \hat{\overline{\theta}}(A)$ and $y \in \hat{\overline{\theta}}(B)$. Then $x + a_1 \in \overline{\theta}(A)$ and $y + b_1 \in \overline{\theta}(B)$ for some $a_1 \in \overline{\theta}(A)$ and $b_1 \in \overline{\theta}(B)$. Now

$$(x + a_1)(y + b_1) + a_1b_1 = xy + (a_1y + a_1b_1) + ((xb_1 + a_1b_1))$$
$$= xy + a_1(y + b_1) + (x + a_1)b_1 \in \overline{\theta}(A)\overline{\theta}(B).$$

This implies $xy \in \overline{\theta}\widehat{(A)\overline{\theta}}(B)$, because $a_1(y + b_1), (x + a_1)b_1 \in \overline{\theta}(A)\overline{\theta}(B)$. Thus $\hat{\overline{\theta}}(A)\hat{\overline{\theta}}(B) \subseteq \overline{\theta}\widehat{(A)\overline{\theta}}(B)$ and hence $\overline{\theta}\widehat{(A)\overline{\theta}}(B) \subseteq \overline{\theta}\widehat{\widehat{(A)\overline{\theta}}}(B)$

$$= \overline{\theta}\widehat{(A)\overline{\theta}}(B)$$

Thus $\overline{\theta}\widehat{(A)\overline{\theta}}(B) = \hat{\overline{\theta}}\widehat{(A)\hat{\overline{\theta}}}(B)$.

Lemma 3.6. *Let θ be congruence relation on R and I be a subset of R. If I is a k-ideal(resp. left k-ideal, right k-ideal) of R if and only if $\overline{\theta}(I) = \hat{\theta}(I)$.*

Proof. Let I be a k-ideal of R. Then by Theorem 3.3, $\theta(I)$ is a rough k-ideal of R. Then both $\overline{\theta}(I)$ and $\underline{\theta}(I)$ are k-ideals of R. Clearly $\overline{\theta}(I) \subseteq \hat{\overline{\theta}}(I)$. Let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$. Since $a, a + x \in \overline{\theta}(I)$, $a \in \overline{\theta}(I)$. Hence $a \in \overline{\theta}(I)$ and so $\hat{\overline{\theta}}(I) \subseteq \overline{\theta}(I)$. Thus $\overline{\theta}(I) = \hat{\overline{\theta}}(I)$.

Similarly $\underline{\theta}(I) = \hat{\underline{\theta}}(I)$. Thus

$$\theta(I) = (\underline{\theta}(I), \overline{\theta}(I))$$
$$= \left(\hat{\underline{\theta}}(I), \hat{\overline{\theta}}(I)\right)$$
$$= \hat{\theta}(I)$$

Conversely, assume that $\theta(I) = \hat{\theta}(I)$, by Lemma 3.4, $\widehat{\theta(I)}$ is a rough k-ideal of R. Thus $\theta(I)$ is rough k-ideal of R. By Theorem 3.3, I is a k-ideal of R.

Lemma 3.7 *Let θ be congruence relation on R. For any two subsets I, J of R with $I \subseteq J$ implies $\hat{\theta}(I) \subseteq \hat{\theta}(J)$.*

Proof: By Theorem 2.9(vi), we have $\overline{\theta}(I) \subseteq \overline{\theta}(J)$ and $\underline{\theta}(I) \subseteq \underline{\theta}(J)$. Let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$. This implies that $a + x \in \overline{\theta}(I) \subseteq \overline{\theta}(J)$ and for some $x \in \overline{\theta}(I) \subseteq \overline{\theta}(J)$. Thus $a \in \hat{\overline{\theta}}(J)$ and so $\hat{\overline{\theta}}(I) \subseteq \hat{\overline{\theta}}(J)$. Similarly $\hat{\underline{\theta}}(I) \subseteq \hat{\underline{\theta}}(J)$ can also be proved $\hat{\theta}(I) \subseteq \hat{\theta}(J)$.

Lemma 3..8. *Let θ be any congruence relation on R and A and B be any subsets of R. If A and B are respectively, right and left k-ideals of R then*

(i) $\overline{\theta}\widehat{(A)\overline{\theta}}(B) \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$.

(ii) $\underline{\theta}\widehat{(A)\,\underline{\theta}}(B) \subseteq \underline{\theta}(A) \cap \underline{\theta}(B)$.

Proof. Let A and B right and left k-ideals or R. By Theorem 3.3, $\overline{\theta}(A)$ and $\overline{\theta}(B)$ are right and left k-ideals of R. Let $x \in \overline{\theta}\widehat{(A)\overline{\theta}}(B)$. Then $x + a_1b_1 \in \overline{\theta}(A)\overline{\theta}(B)$ for some $a_1 \in \overline{\theta}(A)$ and $b_1 \in \overline{\theta}(B)$. Since $\overline{\theta}(A)$ is a right ideal , $a_1b_1, x + a_1b_1 \in \overline{\theta}(A)$ and hence $x \in \overline{\theta}(A)$, *as* $\overline{\theta}(A)$ is a right k-ideal of R. Similarly $x \in \overline{\theta}(B)$ and so $x \in \overline{\theta}(A) \cap \overline{\theta}(B)$. Therefore $\overline{\theta}\widehat{(A)\overline{\theta}}(B) \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$.

(ii) It is similar to (i), hence omitted.

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ISSN: 2278-6848 | Volume: 05 Issue: 01 | March 2014
Paper is available at www.jrps.in | Email : info@jrps.in

Remark 3.9. The reverse inclusions in (i) and (ii) above does not hold.

Theorem 3.10. *Let θ be a relation and I, J be subsets of R.*

(i) *If I and J are left ideals of R, then $(\theta(I):\theta(I))_l$ is a rough ideal of .*

(ii) *If I and J are right ideals of R, then $(\theta(I):\theta(I))_r$ is rough ideal of R.*

Proof. (i) By Theorem 2.11 and Theorem 2.12, we have $\overline{\theta}(I), \overline{\theta}(J), \underline{\theta}(I)$ and $\underline{\theta}(J)$ are left ideals of . Let $x_1, x_2 \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Now $x_1\overline{\theta}(J) \subseteq \overline{\theta}(I)$ and $x_2\overline{\theta}(J) \subseteq \overline{\theta}(I)$. Let $j \in \overline{\theta}(J)$. Then $(x_1 + x_2)j = x_1 j + x_2 j \in \overline{\theta}(I)$. Thus $x_1 + x_2 \in \left(\overline{\theta}(I), \overline{\theta}(J)\right)_l$. Again let $x \in \left(\overline{\theta}(I), \overline{\theta}(J)\right)_l$. For any $r \in R$ and $j \in \overline{\theta}(I), x(rj) = (xr)j \in xr\overline{\theta}(J) \subseteq \overline{\theta}(I)$. Next $(rj)x = (rx)j \in rx\overline{\theta}(J) \subseteq \overline{\theta}(J)$. Thus rx and $xr \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Hence $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is an ideal of R.

Similarly $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_l$ is an ideal of R. Thus $(\theta(I):\theta(J))_l$ is a rough ideal of R.

(ii) The proof is similar to (i), hence it is omitted.

Theorem 3.11. *Let θ be a congruence relation and I, J be subsets of R.*

(i) *If I is a left k-ideals and J is a left ideal of R,* then $(\theta(I):\theta(J))_l$ *is rough k-ideal of R.*

(ii) *If I is a right k-ideal and J is a right ideal then $(\theta(I):\theta(J))_r$ is a rough k-ideal of R.*

Proof. By Theorem 3.10, $(\theta(I):\theta(J))_l$ is a rough ideal of R. This means that $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ and $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_l$ are ideals of R. Suppose $x, x + y \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Then $x\overline{\theta}(J) \subseteq \overline{\theta}(I)$ and $(x + y)\overline{\theta}(J) \subseteq \overline{\theta}(I)$. Thus $xb \in \overline{\theta}(I)$ and $xb + yb \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is a k-ideal, $yb \in \overline{\theta}(I)$. Thus $y \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ which implies that $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is a k-ideal of R. In a similar way, one can show that $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_l$ is a k-ideal of R.

Thus $(\theta(I):\theta(J))_l$ is a rough k-ideal of .

(ii) The proof is very similar to (i), hence omitted.

Lemma 3.12. *Let θ be a congruence relation on R.*

(i) *I is a k-ideal of R if and only if $\overline{\theta}(I)$ is k-ideal of R.*

(ii) *I is a k-ideal of R with $\underline{\theta}(I)$ is nonempty if and only if $\underline{\theta}(I)$ is k-ideal of R.*

Proof. (i) Let I be a k-ideal of R. Let $a, a + b \in \overline{\theta}(I)$. Then $[a]_\theta \cap I \neq \emptyset$ and $[a + b]_\theta \cap I \neq \emptyset$. This implies that there exists $x, y \in R$ such that $x \in [a]_\theta, x \in I$ and $y \in [a + b]_\theta$ and $y \in I$. Now $x \in [a]_\theta$ implies $a \in [x]_\theta \subseteq I, a \in I$ and $y \in [a + b]_\theta$ implies $a + b \in [y]_\theta \subseteq I, a + b \in I$. I being k-ideal, $b \in I$. Thus $[b]_\theta \cap I \neq \emptyset$ and this shows that $b \in \overline{\theta}(I)$ and thus $\overline{\theta}(I)$ k-ideal of R. Similarly, $\underline{\theta}(I)$ is a k-ideal of . Thus $\theta(I)$ is a rough k-ideal of R.

Conversely, assume that $a, a + b \in I, a, b \in R$. Then $[a]_\theta \cap I \neq \emptyset$ and $[a + b]_\theta \cap I \neq \emptyset$. This means that, $+b \in \overline{\theta}(I)$. Thus $b \in \overline{\theta}(I)$. Then there exists $x \in R$ such that $x \in I$ and $x \in [b]_\theta$. This implies that $b \in [x]_\theta \subseteq I$, hence $b \in I$. Thus I is a k-ideal of R.

(ii) Let $a, a + b \in \underline{\theta}(I)$. Then $[a]_\theta \subseteq I$ and $[a + b]_\theta \subseteq I$. This implies that $a, a + b \in I$. Since I is a k-ideal of R, $b \in I$ and $[b]_\theta \subseteq I$. Thus $\underline{\theta}(I)$ k-ideal of R.

Conversely, assume that $\underline{\theta}(I)$ is a k-ideal of R. Let $a, a + b \in I$. Then $[a]_\theta \subseteq I$ and $[a + b]_\theta \subseteq I$. This means that, $a + b \in \underline{\theta}(I)$. This shows that $b \in \underline{\theta}(I)$. Thus $[b]_\theta \subseteq I$ and hence $b \in I$. Thus I is a k-ideal of R.

Lemma 3.13. *Let θ be congruence relation on R and I be a subset of R. If $\overline{\theta}(I)$ is an ideal of R, then $\hat{\overline{\theta}}(I)$ is a k-ideal of R.*

Proof . Let $a_1, a_2 \in \hat{\overline{\theta}}(I)$. Now $a_1 + x_1, a_2 + x_2 \in \overline{\theta}(I)$ for some $x_1, x_2 \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is an ideal , we have $(a_1 + a_2) + (x_1 + x_2) \in \overline{\theta}(I)$ where $x_1 + x_2 \in \overline{\theta}(I)$ and so $a_1 + a_2 \in \hat{\overline{\theta}}(I)$. Hence $\hat{\overline{\theta}}(I)$ is closed under addition. Let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$.
For any $r \in R$, since $\overline{\theta}(I)$ is an ideal of R, $(a + x)r \in \overline{\theta}(I)$ and so $ar + xr \in \overline{\theta}(I)$ for some $xr \in \overline{\theta}(I)$. Thus $ar \in \hat{\overline{\theta}}(I)$. In a similar way, it can be proved that $ra \in \hat{\overline{\theta}}(I)$. Let $a, a + b \in \hat{\overline{\theta}}(I)$. Now $a + x, (a + b) + y \in \overline{\theta}(I)$ for some $x, y \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is an ideal,
$((a + b) + y) + x \in \overline{\theta}(I)$ implies $b + [(a + x) + y] \in \overline{\theta}(I)$. Since $[(a + x) + y] \in \overline{\theta}(I)$, we have $b \in \hat{\overline{\theta}}(I)$. Thus $\hat{\overline{\theta}}(I)$ is a k-ideal of R.

Lemma 3.14. *Let θ be congruence relation on R and I be a subset of R with $\underline{\theta}(I)$nonempty. If $\underline{\theta}(I)$ is an ideal of R, then $\hat{\underline{\theta}}(I)$ is a k-ideal of R.*

Proof . Let $a_1, a_2 \in \hat{\underline{\theta}}(I)$. Now $a_1 + x_1, a_2 + x_2 \in \underline{\theta}(I)$ for some $x_1, x_2 \in \underline{\theta}(I)$. Since $\underline{\theta}(I)$ is an ideal , we have $(a_1 + a_2) + (x_1 + x_2) \in \underline{\theta}(I)$ where $x_1 + x_2 \in \underline{\theta}(I)$ and so $a_1 + a_2 \in \hat{\underline{\theta}}(I)$. Hence $\hat{\underline{\theta}}(I)$ is closed under addition. Let $a \in \hat{\underline{\theta}}(I)$. Then $a + x \in \underline{\theta}(I)$ for some $x \in \underline{\theta}(I)$.
For any $r \in R$,since $\underline{\theta}(I)$ is an ideal of R, $(a + x)r \in \underline{\theta}(I)$ and so $ar + xr \in \underline{\theta}(I)$ for some $xr \in \underline{\theta}(I)$. Thus $ar \in \hat{\underline{\theta}}(I)$. In a similar way, it can be proved that $ra \in \hat{\underline{\theta}}(I)$. Let $a, a + b \in \hat{\underline{\theta}}(I)$. Now $a + x, (a + b) + y \in \underline{\theta}(I)$ for some $x, y \in \underline{\theta}(I)$. Since $\underline{\theta}(I)$ is an ideal , $((a + b) + y) + x \in \underline{\theta}(I)$ implies $b + [(a + x) + y] \in \underline{\theta}(I)$. Since $[(a + x) + y] \in \underline{\theta}(I)$, we have $b \in \hat{\underline{\theta}}(I)$. Thus $\hat{\underline{\theta}}(I)$ is a k-ideal of R.

Lemma 3.5.15. *Let θ be congruence relation on R and I be a subset of R.*

- *(i) Let $\overline{\theta}(I)$ is an ideal of semiring R. $\overline{\theta}(I)$ is a k-ideal if and only if $\overline{\theta}(I) = \hat{\overline{\theta}}(I)$.*
- *(ii) Let $\underline{\theta}(I)$ is an ideal of semiring R. $\underline{\theta}(I)$ is a k-ideal if and only if $\underline{\theta}(I) = \hat{\underline{\theta}}(I)$.*

Proof. Assume that $\overline{\theta}(I)$ is a k-ideal of R. Clearly $\overline{\theta}(I) \subseteq \hat{\overline{\theta}}(I)$. Let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$. Since $x, a + x \in \overline{\theta}(I)$, and $\overline{\theta}(I)$ is a k-ideal of R, $a \in \overline{\theta}(I)$. Hence $a \in \overline{\theta}(I)$ and so $\hat{\overline{\theta}}(I) \subseteq \overline{\theta}(I)$. Thus $\overline{\theta}(I) = \hat{\overline{\theta}}(I)$.
Conversely, let us assume that $\overline{\theta}(I) = \hat{\theta}(I)$, by Lemma 3.4, $\widehat{\overline{\theta}(I)}$ is a k-ideal and so $\overline{\theta}(I)$ k-ideal of R.
(ii)Proof is very similar to (i), hence omitted.

Lemma 3.16. *Let θ be congruence relation on R. For any two subsets I, J of R with $I \subseteq J$ implies that $\hat{\overline{\theta}}(I) \subseteq \hat{\overline{\theta}}(J)$ and $\hat{\underline{\theta}}(I) \subseteq \hat{\underline{\theta}}(J)$.*

Proof: By Theorem 2.9(vi), we have $\overline{\theta}(I) \subseteq \overline{\theta}(J)$ and $\underline{\theta}(I) \subseteq \underline{\theta}(J)$. Let $a \in \hat{\overline{\theta}}(I)$. Then $a + x \in \overline{\theta}(I)$ for some $x \in \overline{\theta}(I)$. This implies that $a + x \in \overline{\theta}(I) \subseteq \overline{\theta}(J)$ and for some $x \in \overline{\theta}(I) \subseteq \overline{\theta}(J)$. Thus $a \in \hat{\overline{\theta}}(J)$ and so $\hat{\overline{\theta}}(I) \subseteq \hat{\overline{\theta}}(J)$. Similarly one can prove that $\hat{\underline{\theta}}(I) \subseteq \hat{\underline{\theta}}(J)$.

Theorem 3.17. *Let θ be a congruence relation and I, J be subsets of R.*

- (i) *If $\overline{\theta}(I)$* and *$\overline{\theta}(J)$ are left-ideals of R, then $\left(\overline{\theta}(I) : \overline{\theta}(J)\right)_l$ is an ideal of R.*
- (ii) *If $\underline{\theta}(I)$*and *$\underline{\theta}(J)$ are left-ideals of R with $\underline{\theta}(I)$* and *$\underline{\theta}(J)$ nonempty subsets of R, then $\left(\underline{\theta}(I) : \underline{\theta}(J)\right)_l$ is an ideal of R.*

Proof. (i) Let $x_1, x_2 \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Now $x_1\overline{\theta}(J) \subseteq \overline{\theta}(I)$ and $x_2\overline{\theta}(J) \subseteq \overline{\theta}(I)$. Let $j \in \overline{\theta}(J)$. Hence $(x_1 + x_2)j = x_1 j + x_2 j \in \overline{\theta}(I)$. Thus $x_1 + x_2 \in \left(\overline{\theta}(I),\overline{\theta}(J)\right)_l$. Again let $x \in \left(\overline{\theta}(I),\overline{\theta}(J)\right)_l$. For any $r \in R$ and $j \in \overline{\theta}(I), x(rj) = (xr)j \in xr\overline{\theta}(J) \subseteq \overline{\theta}(I)$. Next $(rj)x = (rx)j \in rx\overline{\theta}(J) \subseteq \overline{\theta}(J)$. Thus rx and $xr \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Hence $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is an ideal of R.
(ii)It is similar to (i).

Remark 3.18. As the condition of the theorem are only necessary the concept of converse does not arise.

Thorem 3.19. *Let θ be a congruence relation*

(i) *If $\overline{\theta}(I)$ is a left k-ideals and $\overline{\theta}(J)$ is a left ideal of R, then $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is a k-ideal of R.*

(ii) *If $\underline{\theta}(I)$ is a right k-ideal and $\underline{\theta}(J)$ is a right ideal , then $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_r$ is a k-ideal of R.*

Proof. (i) By Theorem 3.10, $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is an ideal of R. It is enough to prove that $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ k-ideal of R. Suppose $x, x + y \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Then $x\overline{\theta}(J) \subseteq \overline{\theta}(I)$ and $(x + y)\overline{\theta}(J) \subseteq \overline{\theta}(I)$. This implies that for any $b \in \overline{\theta}(J)$, $xb \in \overline{\theta}(I)$. Now $xb \in \overline{\theta}(I)$ and $xb + yb \in \overline{\theta}(I)$. Since $\overline{\theta}(I)$ is a k-ideal ,we have $yb \in \overline{\theta}(I)$. This shows that $y \in \left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$. Thus $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_l$ is a k-ideal of R.
(ii)The proof is similar to (i), hence it is omitted.
The following Theorem is similar to Theorem 3.10

Theorem 3..20. *Let θ be a congruence relation and I, J be any subsets of R.*

(i) *If $\overline{\theta}(I)$ and $\overline{\theta}(J)$are right-ideals of R, then $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_r$ is an ideal of R.*

(ii) *If $\underline{\theta}(I)$ and $\underline{\theta}(J)$ are right-ideals of R with $\underline{\theta}(I)$ and $\underline{\theta}(J)$ nonempty subsets of R, then $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_r$ is a ideal of R.*

The following Theorem is similar to Theorem 3.11.

Theorem 3..21. *Let θ be a congruence relation and I, J be any subsets of R.*

(i) *If $\overline{\theta}(I)$ is a right-k-ideal and $\overline{\theta}(J)$ is a right ideal of R, then $\left(\overline{\theta}(I):\overline{\theta}(J)\right)_r$ is a k-ideal of R.*

(ii) *If $\underline{\theta}(I)$ is a right-k-ideal and $\underline{\theta}(J)$ is a right ideal of R, then $\left(\underline{\theta}(I):\underline{\theta}(J)\right)_r$ is a k-ideal of R.*

4. Conclusion

The purpose of this research work is to make contribution to the theoretical development of rough sets as applied to the algebraic structure semiring. Using congruence relation we have studied the properties of the approximations of different types of ideals in semirings and regular semirings. These developments can be considered as theoretical applications of rough sets. Once a data with the algebraic structure studied in this paper is identified, it will not be difficult to develop practical applications of the theory in Knowledge Discovery in Databases(KDD), Data Mining, Pattern recognition or Data classification.

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