© INTERNATIONAL JOURNAL FOR RESEARCH PUBLICATION & SEMINAR
ISSN: 2278-6848 | Volume: 10 Issue: 01 | January - March 2019
Paper is available at www.jrps.in | Email : info@jrps.in

A Study on Curvature in Lorentzian Generalized Sasakian-Space-Forms and Its Applications

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Doi : https://doi.org/10.36676/jrps.v10.i1.1630

Abstract
In this paper, we investigate the curvature properties of Lorentzian generalized Sasakian-space-forms. We establish the necessary and sufficient conditions for these manifolds to be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric, exploring their interrelationships. Additionally, as an application of these theorems, we study the behavior of Ricci almost solitons on conformally flat Lorentzian generalized Sasakian-space-forms.
Key words: Lorentzian, Sasakian-Space-Forms, Curvature, semisymmetric, Ricci, Differential Geometry

1. Introduction
Gauge theory, as we all know, has a lot of profound intension and it has permeated all aspects of theoretical physics. It will surely guide future developments in theoretical physics. Gauge theory and principal fiber bundle theory are inextricably linked with each other (see [1]). For instance, the field strength $f^{\kappa}_{\mu\nu}$ of gauge theory is exactly the curvature of a manifold (see [2]). So if we know the curvature properties of a manifold, we can get the distribution of field strength $f^{\kappa}_{\mu\nu}$. The purpose of our paper is to clarify the unsteady field around Lorentzian generalized Sasakian-space-forms in view of principal fiber bundle theory.
In differential geometry, the curvature tensor R is very significant to the nature of a manifold. Many other curvature tensor fields defining on the manifold are related with curvature tensor, for instance, Ricci tensor S, scalar curvature r, and conharmonic curvature tensor K. It has been proven that the curvature depends on sectional curvatures entirely. If a manifold is of constant sectional curvature, then we call it a space-form.
For a *Sasakian manifold*, we have the definition of ϕ-sectional curvature and it plays the same role as a sectional curvature. If the ϕ-sectional curvature of a Sasakian

manifold is constant, then the manifold is a *Sasakian-space-form* (see [3]). As a generalization of Sasakian-space-form,*generalized Sasakian-space-form* was introduced and investigated in [4] and the authors also gave some examples. In short, a generalized Sasakian-space-form is an almost contact metric manifold that the curvature tensor R is related with three smooth functions f_1, f_2, and f_3 defined on the manifold.

In [5], the authors defined the *generalized indefinite Sasakian-space-form*. It is the generalized Sasakian-space-form with a semi-Riemannian metric. In this paper, we are most interested in the Lorentzian manifold because it is very useful in Einstein's general relativity. In *Lorentzian generalized Sasakian-space-form*, and to make our paper more concise, we will write it as LGSSF for short. We give the necessary and sufficient condition of the LGSSF with the dimension equal to or greater than five to be some certain curvature tensor conditions. This article also clarify the necessary and sufficient condition that LGSSF is *Ricci semisymmetric*. It is meaningful to dig into LGSSF satisfying these conditions because we can understand the relationship between the functions f_1, f_2, and f_3 and the curvature properties of the manifold.

Ricci flow is a powerful tool to investigate manifolds. It was first introduced by Hamilton in [6], and he used it to investigate Riemannian manifolds with positive curvature. There are many solutions to Ricci flow, and the *Ricci soliton* is the self-similar solution of it. Physicists are also interested in the Ricci soliton because in physics, it is regarded as a quasi-Einstein metric. In our paper, we give the Ricci soliton equation as follows:

$$L_W g + 2S = 2\lambda g \tag{1}$$

In the equation, L_W denotes the Lie derivative, S denotes the Ricci tensor, g denotes the Riemannian metric, and λ is a real scalar. We call it the triple (g, W, and λ) Ricci soliton on the manifold. People can also use the Ricci soliton to study semi-Riemannian manifolds and refer to [7–9] for more details.

In [10], Pigola et al. introduced and studied the *Ricci almost soliton*. They replaced the real scalar λ by a smooth function defining the manifold and called it the triple (g, W, and λ) Ricci almost soliton. In our paper, we apply the Ricci almost soliton to LGSSF, and in consideration of the curvature properties of the manifolds, we get some interesting results.

We organize our paper as follows. In Section 2, readers can get several basic definitions about LGSSF. Sections 3, 4, 5, and 6 are dedicated to showing how a LGSSF can be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric. In Section 7, we apply what we get from Sections 3, 4, 5, and 6 to a Ricci almost soliton on LGSSF and give two examples.

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ISSN: 2278-6848 | Volume: 10 Issue: 01 | January - March 2019
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We use U, W, V, X, Y, and Z to denote the smooth tangent vector fields on the manifold, and all manifolds and functions mentioned in paper are smooth.

2. Preliminaries

If a semi-Riemannian manifold M admits a vector field ζ (we call it a *Reeb vector field* or *characteristic vector field*), a 1-form η, and a (1,1) tensor field ϕ satisfying

(2)

where $\varepsilon = g(\zeta, \zeta) = \pm 1$, then call such a manifold an ε-almost contact metric manifold [11] or almost contact pseudometric manifold [12], and call it the triple (ϕ, ζ, and η) almost contact structure on the manifold.

If the 2-form $d\eta$ and the metric g satisfy.

$$d\eta (U, W) = g(U, \phi W), \tag{3}$$

Then the manifold M is a contact pseudometric manifold and the triple (ϕ, ζ, and η) is a contact structure on the manifold.

To define a vector field on the product $\mathbb{R} \times M^{2n+1}$ by (h (d/dx), U); x is the coordinate on $\mathbb{R}$ and h is a C^∞ function on $\mathbb{R} \times M^{2n+1}$. Then define an almost complex structure J on $\mathbb{R} \times M^{2n+1}$ by

$$J\left(h\frac{d}{dx}, U\right) = \left(\eta(U)\frac{d}{dx}, \phi U - h\zeta\right), \tag{4}$$

$$\begin{aligned} \phi\zeta &= 0, \\ \eta \circ \phi &= 0, \\ \phi^2 &= -id + \eta \otimes \zeta, \\ \eta(\zeta) &= 1, \\ \eta(U) &= \varepsilon g(\zeta, U), \\ g(U, W) &= g(\phi U, \phi W) + \varepsilon \eta(U) \eta(W), \end{aligned}$$

and it is easy to check $J^2 = -id$. Moreover, if J is integrable, then will say the almost contact structure (ϕ, ζ, and η) is normal (see [3]). Call an ε-normal contact metric manifold an indefinite Sasakian manifold or an ε-Sasakian manifold. Now give the definition of the ϕ-sectional curvature. The plane spanned by U and ϕU is called ϕ-section if U is orthogonal to ζ. The ϕ-sectional curvature is the sectional curvature $K(U, \phi U)$. The curvature of an indefinite Sasakian manifold is determined by ϕ-sectional curvatures entirely. If the ϕ-sectional curvature of an ε-Sasakian manifold is a constant c, then the curvature tensor of the manifold has the following form [13]:

$$\begin{aligned} R(U, W)X = {} & \frac{c + 3\varepsilon}{4}\{g(W, X)U - g(U, X)W\} \\ & + \frac{c - \varepsilon}{4}\{g(U, \phi X)\phi W - g(W, \phi X)\phi U + 2g(U, \phi W)\phi X\} \\ & + \frac{c - \varepsilon}{4}\{\eta(U)\eta(X)W - \eta(W)\eta(X)U + \varepsilon g(U, X)\eta(W)\zeta - \varepsilon g(W, X)\eta(U)\zeta\}. \end{aligned} \tag{5}$$

In [5], the author

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ISSN: 2278-6848 | Volume: 10 Issue: 01 | January - March 2019
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replaced the constants with three smooth functions defining the manifold. For an ε-almost contact metric manifold M, if the curvature tensor is given by

$$R(U,W)X = f_1\{g(W,X)U - g(U,X)W\} + f_2\{g(U,\phi X)\phi W - g(W,\phi X)\phi U + 2g(U,\phi W)\phi X\} + f_3\{\eta(U)\eta(X)W - \eta(W)\eta(X)U + \varepsilon g(U,X)\eta(W)\zeta - \varepsilon g(W,X)\eta(U)\zeta\}, \quad (6)$$

where $f_1, f_2, f_3 \in C^\infty(M)$, then call M the generalized indefinite Sasakian-space-form.

In this paper, the only focus is on the Lorentzian situation: $\varepsilon = -1$ and the index of the metric is one. And called such manifold the Lorentzian generalized Sasakian-space-form, and in our paper, we denote it by $M_1^{2n+1}(f_1, f_2, f_3)$. Because some of the curvature tensor fields studied are not suitable for three manifolds, in the following, the dimension of LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is greater than three, that is, $n > 1$

For a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$, we have two useful equations from (**6**):

$$R(U,W)\zeta = (f_1 + f_3)(\eta(U)W - \eta(W)U), \quad (7)$$

$$R(\zeta,U)W = (f_1 + f_3)(g(U,W)\zeta + \eta(W)U). \quad (8)$$

Lemma 1. For a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$, the Ricci tensor S is

$$S(U,W) = (2nf_1 + 3f_2 + f_3)g(U,W) + (3f_2 - (2n-1)f_3)\eta(U)\eta(W), \quad (9)$$

so the Ricci operator Q and scalar curvature r are

$$QU = (2nf_1 + 3f_2 + f_3)U + ((2n-1)f_3 - 3f_2)\eta(U)\zeta, \quad (10)$$

$$r = 2n(2n+1)f_1 + 6nf_2 + 4nf_3. \quad (11)$$

Proof. As we all know for a semi-Riemannian manifold of dimension n, the Ricci tensor S and the scalar curvature r are

$$S(U,W) = \sum_{i=1}^{n} \varepsilon_i g(R(U,E_i)E_i, W),$$

$$r = \sum_{i=1}^{n} \varepsilon_i S(E_i, E_i), \quad (12)$$

where $\{E_i, \cdots, E_n\}$ is a local orthonormal frame field on the manifold and ε_i is the signature of E_i. The curvature tensor of $M_1^{2n+1}(f_1, f_2, f_3)$ is given by $g(U, W) = \sum\varepsilon_i g(U, E_i)g(X, E_i)$, so it is can easily get (**9**), (**10**), and (**11**).

We can use warped product to construct LGSSF (see [5]). Let $h > 0$ be a function on $\mathbb{R}$ and $(N^{2n}, J,$ and $G)$ be an almost complex manifold. Then, the warped product $M = \mathbb{R}\times_h N$ is a LGSSF with the Lorentzian metric given by

$$g_h = -\pi^* (g_{\mathbb{R}}) + (h \circ \pi)^2 \sigma^* (G), \quad (13)$$

where π is the projection from $\mathbb{R} \times N$ to $\mathbb{R}$ and σ is the projection to N. The almost contact structure is

$$\begin{aligned} \zeta &= \frac{\partial}{\partial x}, \\ \eta (U) &= -g_h (U, \zeta), \\ \phi (U) &= (J \sigma_* U)^* . \end{aligned} \quad (14)$$

***Theorem 2* ([5].)** Given a generalized complex space-form $N^{2n}(F_1, F_2)$. Then, is LGSSF, with functions

$$\begin{aligned} f_1 &= \frac{(F_1 \circ \pi) + h'^2}{h^2}, \\ f_2 &= \frac{F_2 \circ \pi}{h^2}, \\ f_3 &= -\frac{(F_1 \circ \pi) + h'^2}{h^2} + \frac{h''}{h}. \end{aligned} \quad (15)$$

3. Projectively Flat Lorentzian Generalized Sasakian-Space-Form

For a $(2n + 1)$-dimensional ($n > 1$) smooth manifold M, the *projective curvature tensor* P is defined by.

$$P (U, W) X = \frac{1}{2n} \{S (U, X) W - S (W, X) U\} + R (U, W) X. \quad (16)$$

It is a way to measure whether a manifold is a space-form because if M is projectively flat ($P = 0$), then it must be of constant curvature and the converse is also true. For more details, readers can refer to [14].

Theorem 3. A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat if and only if $f_2 = f_3 = 0$.

Proof. Firstly, suppose that $P(U, W)X = 0$. Put $U = \zeta$ and replace X by ϕX, then equation (16) will be

$$P (\zeta, W) \phi X = \frac{1}{2n} ((2n - 1) f_3 - 3 f_2) g (W, \phi X) \zeta = 0. \quad (17)$$

n consideration of $g(W, \phi X) \neq 0$, we have

$$(2n - 1) f_3 - 3 f_2 = 0. \quad (18)$$

Then, equation (9) will be

$$S (W, U) = (2nf_1 + 3 f_2 + f_3) g (W, U) = 2n (f_1 + f_3) g (W, U) . \quad (19)$$

By the above equation, can be written as (16)

$$g(P(U,W)X,Z) = f_2\{g(U,\phi X)g(\phi W,Z) - g(W,\phi X)g(\phi U,Z) + 2g(U,\phi W)g(\phi X,Z)\}$$
$$- f_3\{\eta(W)\eta(X)g(U,Z) - \eta(U)\eta(X)g(W,Z) + \eta(W)\eta(Z)g(U,X) - \eta(U)\eta(Z)g(W,X)$$
$$+ g(W,X)g(U,Z) - g(U,X)g(W,Z)\} = 0. \quad (20)$$

Setting $U = \phi U$ and $W = \phi W$, we have

$$g(P(\phi U,\phi W)X,Z)$$
$$= f_2\{g(\phi U,\phi X)g(\phi^2 W,Z) + 2g(\phi U,\phi^2 W)g(\phi X,Z) - g(\phi W,\phi X)g(\phi^2 U,Z)\}$$
$$+ f_3\{g(\phi U,X)g(\phi W,Z) - g(\phi W,X)g(\phi U,Z)\} = 0. \quad (21)$$

Let us denote the orthonormal local basis of TM by $\{e_1, \cdots, e_{2n}, e_{2n+1} = \zeta\}$. Obviously, the signature of the local basis is $\{+, \cdots, +, -\}$ and denote it by $\{\varepsilon_1, \cdots, \varepsilon_{2n}, \varepsilon_{2n+1}\}$.
Putting $W = e_i$ and $Z = \varepsilon_i e_i$ in the above equation and summing over i, the following equation:

$$(f_3 - (2n+1)f_2)\, g(\phi U,\phi X) = 0, \quad (22)$$

since , $g(\phi U,\phi X) = \sum_{i=1}^{2n+1} \varepsilon_i g(\phi U, e_i) g(\phi X, e_i)$.

Because of $g(\phi U, \phi X) \neq 0$, we get

$$f_3 - (2n+1)f_2 = 0. \quad (23)$$

Taking consideration of $(2n - 1)f_3 - 3f_2 = 0$ and $n > 1$, we get

$$f_2 = f_3 = 0. \quad (24)$$

Conversely, we suppose that $f_2 = f_3 = 0$ then use (6) and (9), then (16) will be

$$P(U,W)X = f_1\{g(U,X)W - g(W,X)U\} - f_1\{g(U,X)W - g(W,X)U\} = 0. \quad (25)$$

In order to get the next theorem of our paper, we first introduce the following famous theorem.
Schur.Theorem (see [15]). If $M^n (n \geq 3)$ is a connected semi-Riemannian manifold, and for each $m \in M$, the sectional curvature $K(m)$ is a constant function on the nondegenerate planes in $T_m M$, then $K(m)$ is a constant function on the manifold.

From Theorem 3, we can get if a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is projectively flat, then $K(m) = f_1$. Using Schur.Theorem, we have the following theorem.

Theorem 4. If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat, then f_1 is a constant function.

4. Conharmonically Flat Lorentzian Generalized Sasakian-Space-Form

The conharmonic transformation is a kind of special conformal transformation. In general, a conformal transformation does not preserve the harmonic function defined on the manifold. In [16], Ishii introduced and studied the conharmonic transformation, which preserved a special kind of harmonic function. He also proved that a manifold could be reduced to a flat space by a conharmonic transformation if

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ISSN: 2278-6848 | Volume: 10 Issue: 01 | January - March 2019
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and only if the conharmonic curvature tensor K vanished everywhere on the manifold. In other words, the manifold is conharmonically flat ($K = 0$). For a $(2n + 1)$-dimensional ($n > 1$) smooth manifold, the *conharmonic curvature tensor* K is given by.

$$K(U,W)X = \frac{1}{2n-1}\{g(U,X)QW - g(W,X)QU + S(U,X)W - S(W,X)U\} + R(U,W)X. \tag{26}$$

Definition 7. A $(2n + 1)$-dimensional ($n > 1$) LGSSF is said to be ζ-conharmonically flat if it satisfies

$$K(U,W)\zeta = 0. \tag{27}$$

Lemma 8. A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat if and only if $(2n + 1)f_1 + 3f_2 + 2f_3 = 0$.

Proof. From (7) and (10), equation (27) becomes,

$$K(U,W)\zeta = \frac{1}{2n-1}\{2n(f_1+f_3)\eta(W)U - 2n(f_1+f_3)\eta(U)W + (2nf_1+3f_2+f_3)\eta(W)U$$
$$-(2nf_1+3f_2+f_3)\eta(U)W\} + (f_1+f_3)\{\eta(U)W - \eta(W)U\} = \frac{1}{2n-1}((2n-1)f_1+3f_2$$
$$+2f_3)\{\eta(W)U - \eta(U)W\}. \tag{28}$$

So $M_1^{2n+1}(f_1, f_2, f_3)$ is ζ-conharmonically flat if and only if $(2n + 1)f_1 + 3f_2 + 2f_3 = 0$.

From equation (11) and Lemma 8, we have the following theorem.

Theorem 9. A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat if and only if its scalar curvature $r = 0$.

By Theorem 3 and Lemma 8, we have the following theorem.

Theorem 10. If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat and projectively flat, then it is a flat manifold.

We know that being conharmonically flat is the sufficient condition of ζ-conharmonically flat. So we have the following theorem.

Theorem 11. If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat and projectively flat, then it is a flat manifold.

It is very important for us to know how a LGSSF can be conharmonically flat.

Theorem 12. A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat if and only if $f_2 = 0$ and $(2n + 1)f_1 + 2f_3 = 0$.

Proof. Comparing (26) with we can get

$$C(U,W)X = \frac{(2n-1)f_1+3f_2+2f_3}{2n-1}\{g(W,X)U - g(U,X)W\} + K(U,W)X. \tag{29}$$

If $f_2 = 0$ and $(2n + 1)f_1 + 2f_3 = 0$, then from Theorem 4 .

$$K(U,W)X = C(U,W)X - \frac{(2n+1)f_1+3f_2+2f_3}{2n-1}\{g(W,X)U - g(U,X)W\} = 0. \tag{30}$$

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5. Ricci Semisymmetric Lorentzian Generalized Sasakian-Space-Form:

There are many classes of smooth manifolds such as locally symmetric and Ricci symmetric. A smooth manifold is Ricci semisymmetric when the curvature operator $R(U, W)$ acting on S vanishes identically, that is

$$R(U,W)\cdot S = 0. \tag{31}$$

Theorem 14. A $(2n + 1)$-dimensional $(n > 1)$ LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is Ricci semisymmetric if and only if $f_1 + f_3 = 0$ or $3f_2 = (2n - 1)f_3$.

Proof. First, suppose that $M_1^{2n+1}(f_1, f_2, f_3)$ is Ricci semisymmetric, that is

$$(R(U,W)\cdot S)(Y,Z) = -S(Y,R(U,W)Z) - S(R(U,W)Y,Z) = 0. \tag{32}$$

Put $U = \zeta$ in the above equation, then we will have

$$S(R(\zeta,W)Y,Z) + S(Y,R(\zeta,W)Z) = 0. \tag{33}$$

Then, using (8), we can get

$$(f_1+f_3)\{g(W,Y)S(\zeta,Z)+\eta(Y)S(W,Z)+g(W,Z)S(\zeta,Y)+\eta(Z)S(W,Y)\}$$
$$= (f_1+f_3)((2n-1)f_3-3f_2)\{-2\eta(Y)\eta(W)\eta(Z)-\eta(Z)g(W,Y)-\eta(Y)g(W,Z)\} = 0. \tag{34}$$

Again we use the orthonormal basis $\{e_1, \cdots, e_{2n+1} = \zeta\}$ with signature $\{\varepsilon_1, \cdots, \varepsilon_{2n}, \varepsilon_{2n+1} = \varepsilon\}$, and this time, in the above equation, we suppose $W = e_i$ and $Z = \varepsilon_i e_i (1 \le i \le 2n + 1)$, and taking summation over i, we can get

$$2n(f_1+f_3)((2n-1)f_3-3f_2)\eta(Y) = 0. \tag{35}$$

Hence, we get $f_1 + f_3 = 0$ or $(2n - 1)f_3 - 3f_2 = 0$.

Conversely, if $(2n - 1)f_3 - 3f_2 = 0$, then by direct calculation,

$$(R(U,W)\cdot S)(Y,Z) = -S(Y,R(U,W)Z) - S(R(U,W)Y,Z)$$
$$= -(2nf_1+3f_2+f_3)\{g(R(U,W)Z,Y)+g(R(U,W)Y,Z)\} = 0. \tag{36}$$

Theorem 15. If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat and Ricci semisymmetric, then it is a flat manifold.

Proof. From Theorem 12 and Theorem 14, we know that if a LGSSF is conharmonically flat and Ricci semisymmetric, then we will have $f_2 = 0$, $(2n + 1)f_1 + 2f_3 = 0$, and $3f_2 = (2n - 1)f_3$ or $f_1 + f_3 = 0$. In any case, we get $f_1 = f_2 = f_3 = 0$.

Notice that $f_2 = f_3 = 0$ satisfies $(2n - 1)f_3 - 3f_2 = 0$, so we can get the following theorem.

Theorem 16. If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat, then it is Ricci semisymmetric.

6. Practical application of Curvature Properties on Lorentzian:

Curvature properties on Lorentzian Generalized Sasakian-Space-Forms find practical applications primarily in the field of theoretical physics, particularly in areas like gravity theory, gauge theory, and cosmology due to their ability to model complex spacetime geometries with specific curvature characteristics, allowing researchers to study phenomena like black holes, wormholes, and the early universe under specific conditions.

6.1. Some key applications include:

1. Modeling exotic spacetimes: Lorentzian Generalized Sasakian-Space-Forms provide a framework to study non-standard spacetime geometries with varying curvature properties, which can be useful for theoretical investigations into alternative gravity theories or extreme astrophysical scenarios.
2. Studying gravitational lensing: Analyzing the curvature properties of these spaceforms, researchers can investigate how light bends around massive objects like black holes, providing insights into gravitational lensing phenomena.
3. Exploring particle physics in curved space time: The specific curvature characteristics of these spaces can be used to study how particles behave in curved spacetime, potentially providing insights into quantum gravity theories.
4. Investigating cosmological models: By incorporating the curvature properties of Lorentzian Generalized Sasakian-Space-Forms into cosmological models, researchers can study the evolution of the universe under different conditions.

8. Conclusion

From the above properties it presents the necessary and sufficient conditions for LGSSF to be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric. As a result, the study shows how to construct a Lorentzian manifold with certain curvature tensor conditions, which is useful in gauge theories because of the correspondence between curvature and field strength. It also play a vital role in studying gravitational lensing and Investigating cosmological models and much more.

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