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Stability of Quadratic Functional Equation

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1. Introduction

In 1897, Hensel [1] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [2–5]).

A *valuation* is a function | · | from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function | · | is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function | · | taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1. *Let X be a vector space over a field K with a non-Archimedean valuation* | · |. *A function* $\|\cdot\| : X \to [0, \infty)$ *is said to be a non-Archimedean norm if it satisfies the following conditions:*

(i) $\|x\| = 0$ *if and only if* $x = 0$;

(ii) $\|rx\| = |r|\,\|x\|$ $(r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ *is called a non-Archimedean normed space*.

Definition 1.2. *(i) Let* $\{x_n\}$ *be a sequence in a non-Archimedean normed space X. Then the sequence* $\{x_n\}$ *is called Cauchy if for a given* $\varepsilon > 0$ *there is a positive integer N such that*

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ *be a sequence in a non-Archimedean normed space X. Then the sequence* $\{x_n\}$ *is called convergent if for a given* $\varepsilon > 0$ *there are a positive integer N and an* $x \in X$ *such that*

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. *Then we call* $x \in X$ *a limit of the sequence* $\{x_n\}$, *and denote by* $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y)$, $\langle x, y \rangle = 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Pinsker [6] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [7] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y), \quad x\perp y,$$

in which $\perp$ is an abstract orthogonality relation was first investigated by Gudder and Strawther [8]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [9] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [10] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz; cf. [9].

Suppose X is a real vector space with dim $X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of non-negative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

The first author treating the stability of the quadratic equation was Skof [25] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. Cholewa [26] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by Czerwik [27] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [28–32]).

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x\perp y$$

was first investigated by Vajzović [33] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [34], Fochi [35] and Szabó [36] generalized this result. See also [37].

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [38–51]).

Katsaras [52] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In particular, Bag and Samanta [53], following Cheng and Mordeson [54], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [55]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [56].

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Definition 1.3. (Bag and Samanta [53]) *Let X be a real vector space. A function $N : X \times \mathbb{R} \to 0[1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
($N1$) $N(x, t) = 0$ *for* $t \leq 0$*;*
($N2$) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$*;*
($N3$) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$*;*
($N4$) $N(x + y, c + t) \geq min\{N(x, s), N(y, t)\}$*;*
($N5$) $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$*;*
($N6$) *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$*.*
The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector space and examples of fuzzy norms are given in (see [57, 58]).
Example 1.1. *Let* $(X, \|.\|)$ *be a normed linear space and* $\alpha, \beta > 0$*. Then*

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

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