© INTERNATIONAL JOURNAL FOR RESEARCH PUBLICATION & SEMINAR
ISSN: 2278-6848 | Volume: 02 Issue: 04 | March 2013
Paper is available at www.jrps.in | Email : info@jrps.in

FUNCTIONAL EQUATION

Sonu Bansal

Abstract

The functional equations form a modern branch of mathematics. The origin of functional equations came about the same time as the modern definition of function. During 1747 to 1750, J. d'Alembert published three papers. These three papers were the first on functional equations. Many celebrated mathematicians including S. M. Ulam, Th. M. Rassias, A.L. Cauchy, T. Aoki, F. Skof, D. H. Hyers, J. Bolyai, J. d'Alembert, L. Euler, C. F. Gauss, M. Fr´echet, J.L. Jensen A. N. Kolmogorov, N. I. Lobaceveskkii, J. V. Pexider, S.D. Poisson have studied functional equations because of their apparent simplicity and harmonic nature. In this paper, we investigate the term functional equation and give examples of functional equation.

Functional Equations

The usage of the word functional goes back to the calculus of variation, implying a function whose argument is a function and the name was first used in Hadamard's 1910 book on that subject. However, the general concept of functional had previously been introduced in 1887 by the Italian mathematician and physicist Vito Volterra. The theory of nonlinear functionals was continued by students of Hadamard, in particular Frechet and Levy. Hadamard also founded the modern school of linear functional analysis further developed by Riesz and the group of Polish mathematicians around Stefan Banach.

In functional analysis, a functional is traditionally a map from a vector space to the field underlying the vector space, which is usually the real numbers. In other words, it is a function which takes for its input-argument a vector and returns a scalar. Commonly the vector space is a space of functions, thus the functional takes a function for its input-argument, and then it is sometimes considered a function of a function.

After a formal introduction of concept of functional, we come to the concept of functional equations; we begin with some formal definitions of functional equations as:

- A functional equation is any equation that specifies a function in implicit form. Often the equations relate the value of a function (or functions) at some part with its values at other points. For instance the properties of functions can be determined by considering the types of functional equations they satisfy.
- The term functional equation usually refers to equations that cannot be simply reduced to algebraic equations.
- An equation of the form $f(x, y, z, \ldots) = 0$ where f contains a finite number of independent variables, known functions and unknown functions which are to be solved for.
- Functional equations are equations for unknown functions instead of unknown numbers.
- A functional equation is an equation whose variables are ranging over functions.

Functional equations arising from the problem of vibration of strings, the parallelogram law of forces or rule for addition of vectors were solved by D' Almbert in the paper J. d' Almbert, Memoire sui les principles de mecanique, Hist. Acad. Sci. Paus, 278-286 (1769). Most likely he is the first person to derive and solve functional equations arising from problems of applied mathematics.

Functional equations occur practically everywhere. Their influence and applications are felt in every field, and all fields benefit from their contact, use, and technique. The growth and development used to be influenced by their spectacular applications in several areas not only in mathematics but also in other disciplines. Applications can be found in a wide variety of fields such as analysis, applied sciences, behavioral and social sciences, biology, combinatories, computer, economics, engineering, geometry, inequalities, information theory, inner product spaces, physics, statistics, taxation, etc. Functional equations are being used with vigor in ever-increasing numbers to investigate problems In the above mentioned areas and other fields.

Examples of Functional Equations:

- f(x+y)=f(x)f(y) satisfied by all exponential functions (E)
- f(xy)=f(x)+f(y), satisfied by all logarithmic functions (L)
- f(x+y)=f(x)+f(y), (Cauchy functional equation) (A)
- f(x+y)+f(x-y)=2[f(x)+f(y)] (quadratic functional equation) (Q)
- f((x+y)/2)=(f(x)+f(y))/2 (Jensen equation) (J)
- g(x+y)+g(x-y)=2[g(x)g(y)] (d'Alembert) (D)
- The Riemann zeta function ξ satisfies the following functional equation

$$f(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)f(1-s)$$

where Γ denotes the gamma function.

- These three functional equations are satisfied by the gamma function:

$$f(x) = \frac{f(x+1)}{x},$$

$$f(y)f\left(y+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2y-1}} f(2y)$$

$$f(z)f(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{(Euler's reflection formula)}$$

- Quadratic Jensen functional equations:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \qquad \text{(J1)}$$

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x) \qquad \text{(J2)}$$

© INTERNATIONAL JOURNAL FOR RESEARCH PUBLICATION & SEMINAR
ISSN: 2278-6848 | Volume: 02 Issue: 04 | March 2013
Paper is available at www.jrps.in | Email : info@jrps.in

- Quadratic functional equations: f(x+y)+f(x-y)=2f(x)+2f(y), (Qd1)
  f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), (Qd2)
  f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x), (Qd3)
  f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)=4f(x)+4f(y)+4f(z). (Qd4)

- Cubic functional equations:

  f(x+2y)+f(x-2y)+f(2x)=2f(x)+4f(x+y)+4f(x-y), (C1)

  f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y), (C2)
  f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y), (C3)
  f(x+3y)-3f(x+y)-3f(x-y)-f(x-3y)-48f(y)=0, (C4)
  f(x+y+z)+f(x+y-z)+2f(x)+2f(y)
  =2f(x+y)+f(x+z)+f(x-z)+f(y+z)+f(y-z), (C5)

  f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)
  =2f(x+y)+4f(x+z)+4f(x-z)+4f(y+z)+ 4 f(y-z), (C6)

  f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)+7f(x)+7f(-x)
  =2f(x+y)+4f(x+z)+f(x-z)+f(y+z)+f(y-z), (C7)

  f(x+y+z)+f(x+y-z)+2f(x-y)+4f(y)
  = f(x-y+z)+f(x-y-z)+2f(x+y)+ 2f(y+z)+2f(y-z), (C8)
  f(2x + y) + f(2x – y) = 2f(x +y) +2 f(x –y) + 12f(x) (C9)
  f(3x+y)–3f(x+y)–3f(x–y)–f(3x–y)–48f(x)=0 (C10)
  3f(x+3y)+f(3x+y)–15f(x+y)–15f(x–y)–80f(y)=0 (C11)

- Quartic functional equations:

  $f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$ (Q1)

  f(2x +y) + f(2x –y) =4 f(x +y) +4f(x –y) +24 f(x) - 6f(y), (Q2)
  f(x +2y) + f(x –2y) =2 f(x) +32f(y)+48f(xy), (Q3)
  f(3x +y) + f(x +3y) =24 f(x+y) -6f(x–y) +64 f(x) + 64$\sqrt{f(y)}$ (Q4)

$f\left(\underline{x+y}+z\right)+f\left(\underline{x+y}-z\right)+\left(\underline{x-y}+z\right)+\left(\underline{x-y}-z\right)+f(x)+f(y)$